On the sum of independent zero-truncated Poisson random variables

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Abstract

The objective of this article is to derive the density function and cumulative distribution function for random variables which may be written as the sum of independent (either identical or non-identical) zero-truncated Poisson random variables. The obtained expressions may be particularly useful for modelling purposes, especially in view of linking common purchase quantity models from the marketing literature to stochastic production-inventory models from the operations management literature.

Keywords: Poisson distribution, zero-truncated, probability function

1 Introduction

In this article, we derive the density function and cumulative distribution function for the sum of independent (either identical or non-identical) zero-truncated or positive Poisson random variables (Johnson and Kotz (1969))

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also called conditional Poisson random variables (Cohen (1960)). Though the research presented is theoretical in nature, it is inspired by a very practical objective, namely the desire to link the results of marketing research models to the literature on inventory models, available from the operations management field.

In the marketing literature, modelling the customer's response behavior to price and promotion effects is a major area of research. In this field, it is widely accepted that the consumer's quantity decision (i.e., how many units to buy within a product category) at a given purchase incidence can be modelled as a stochastic variable, following a zero-truncated poisson distribution (e.g., see Bucklin et al (1998); Dillon and Gupta (1996); Silva-Risso et al (1999); Campo et al (2003)). Typically, the Poisson parameter in these models is estimated as a function of customer-specific variables (e.g., loyalty) and marketing variables (e.g., price or promotion).

It is straightforward that the customer's purchase behavior has a direct impact on the availability of finished goods stocks at the retailer, which (through the inventory policy used) triggers replenishment orders either from a warehouse (in multi-echelon systems) or directly from the manufacturing system. As the customer purchase quantity is stochastic, the size of the replenishment order will typically be stochastic too, and will depend on the inventory policy used. In case of a fixed review period policy or an order-up-to policy for example, the size of the replenishment order will consist of a random number of customer order quantities.

Though the integration of inventory models and purchase quantity models would offer vast opportunities for further research development, both fields seem to have evolved separately upto this point. The advanced models in the inventory management literature (e.g., see Dominey and Hill (2004); Hill and Johansen (2004); Matheus and Gelders (2000); Zheng and Federgruen (1991)) assume that the customer's demand process is compound Poisson distributed (Adelson (1996)), assuming that the customer orders arrive according to a Poisson process without making further assumptions on the distribution of customer order size. This paper aims to provide a first step towards the integration of both model types, by determining the probability distribution of a sum of independent zero-truncated poisson random variables. The assumption of independency is justified in our setting, as the purchase quantity decision of a customer is not influenced by the decisions of other customers. To the best of our knowledge, this distribution has not yet been examined. We will study both identically and non-identically distributed zero-truncated Poisson variables. Hence, the resulting expressions permit to reflect both homogenous customer populations (where all customers have the same Poisson purchase rate) and heterogenous customer populations (where purchase rates

among customer classes may differ).
The remainder of the paper is organized as follows: in section 2, we give a brief overview of the characteristics of the zero-truncated Poisson distribution. In section 3, we construct the probability function and cumulative distribution of the sum of independent and identically distributed zero-truncated Poisson distributed random variables. In section 4, the resulting expressions are extended towards non-identically distributed variables. Finally, section 5 summarizes the conclusions and avenues for further research.

2 Characteristics of the zero-truncated Poisson distribution

The density function of a zero-truncated Poisson variable is given by (Johnson and Kotz (1969)):

$$P[X_i = n] = \begin{cases} \dfrac{\lambda^n}{n!(e^\lambda - 1)} & \text{if } n \in I\!N_0 \\ 0 & \text{elsewhere} \end{cases} \tag{1}$$

with parameter $\lambda \in I\!R^+$. The difference with the standard Poisson distribution lies in the correction factor $(1 - e^{-\lambda})^{-1}$, which reflects the fact that a value of 0 cannot occur. The basic parameters such as the mean:

$$\mu_{X_i} = \frac{\lambda e^\lambda}{e^\lambda - 1} \tag{2}$$

and variance

$$\sigma^2_{X_i} = \frac{\lambda e^\lambda}{e^\lambda - 1}\left[1 - \frac{\lambda}{e^\lambda - 1}\right] \tag{3}$$

can easily be derived in a straightforward manner Johnson and Kotz (1969). The higher moments for this type of probability distribution can be obtained from the moment generating function:

$$M_{X_i}(t) = \frac{e^{\lambda e^t} - 1}{e^\lambda - 1} \tag{4}$$

while the cumulative distribution function is given by:

$$F_{X_i}(x) = \begin{cases} \dfrac{e_{\lfloor x \rfloor}(\lambda) - 1}{e^\lambda - 1} & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0 \end{cases} \tag{5}$$

with $\lfloor x \rfloor$ the integer part of x and $e_a(b)$ the exponential sum function defined as (see e.g. Magnus et al (1966))

$$e_a(b) \equiv \sum_{i=0}^{a} \frac{b^i}{i!}, \text{ with } a \in I\!N. \tag{6}$$

3 The sum of identical independent zero-truncated Poisson distributed random variables

In this section, we consider the case of a random variable X which can be expressed as the sum of m independent identical zero-truncated Poisson distributed random variables X_i; $i = 1, \ldots, m$:

$$X = \sum_{i=1}^{m} X_i \tag{7}$$

In order to get insight into the distribution of such a random variable, we first construct the probability function:

$$P[\sum_{i=1}^{m} X_i = n] \tag{8}$$

More specifically, we will prove the following theorem:

Theorem 1 *Let X_i, $i = 1, \ldots, m$ be m independent identical zero-truncated Poisson distributed random variables and let X be the random variable defined as $\sum_{i=1}^{m} X_i$. The probability function for X is given by*

$$P[X = n] = \begin{cases} \dfrac{\lambda^n}{n!(e^\lambda - 1)^m} \displaystyle\sum_{k=0}^{m} (-1)^k (m-k)^n \binom{m}{k} & \text{if } m \leqslant n \in I\!N \\ 0 & \text{elsewhere} \end{cases} \tag{9}$$

Proof: We will prove this theorem by induction. It is clear that the theorem is valid for $m = 1$, yielding the density function given in expression (1). Let us assume that the theorem is valid for $m = M$. Starting from this result, we now demonstrate that the theorem holds for $m = M + 1 \leqslant n$.
As all variables X_i are independent and share the same Poisson parameter λ, we find that

$$\begin{aligned} P[\sum_{i=1}^{M+1} X_i = n] &= \sum_{j=1}^{n-M} P[\sum_{i=1}^{M} X_i = n - j] P[X_{M+1} = j] \\ &= \frac{\lambda^n}{n!(e^\lambda - 1)^{M+1}} \sum_{j=1}^{n-M} \binom{n}{j} \sum_{k=0}^{M} (-1)^k (M-k)^{n-j} \binom{M}{k} \end{aligned}$$

should be equal to

$$\frac{\lambda^n}{n!(e^\lambda - 1)^{M+1}} \sum_{k=0}^{M+1} (-1)^k (M+1-k)^n \binom{M+1}{k}$$

Hence, it suffices to prove that

$$\sum_{k=0}^{M+1} (-1)^k (M+1-k)^n \binom{M+1}{k} = \sum_{j=1}^{n-M} \binom{n}{j} \sum_{k=0}^{M} (-1)^k (M-k)^{n-j} \binom{M}{k} \quad (10)$$

Using Newton's binomium, the l.h.s. of this equation may be rewritten as

$$\begin{aligned}
\text{l.h.s} &= \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{M+1} \binom{M+1}{k} (-1)^k (M-k)^{n-j} \\
&= \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{M+1} \left[\binom{M}{k} + \binom{M}{k-1} \right] (-1)^k (M-k)^{n-j} \\
&= \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{M} \binom{M}{k} (-1)^k (M-k)^{n-j} \\
&\qquad\qquad + \sum_{j=0}^{n} \binom{n}{j} \sum_{k=1}^{M+1} \binom{M}{k-1} (-1)^k (M-k)^{n-j} \\
&= \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{M} \binom{M}{k} (-1)^k (M-k)^{n-j} \\
&\qquad\qquad - \sum_{j=0}^{n} \binom{n}{j} \sum_{l=0}^{M} \binom{M}{l} (-1)^l (M-1-l)^{n-j} \\
&= \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{M} \binom{M}{k} (-1)^k (M-k)^{n-j} - \sum_{l=0}^{M} \binom{M}{l} (-1)^l (M-l)^n
\end{aligned}$$

Now taking into account the following relation (see e.g. Gradshteyn and Ryzhik (2000) formula 0.154.3)

$$\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} k^n = \sum_{k=0}^{m} \binom{m}{k} (-1)^k (m-k)^n \equiv 0, \ \forall n \leqslant m-1 \quad (11)$$

the l.h.s. of eq. (10) reduces to

$$\sum_{j=0}^{n-M} \binom{n}{j} \sum_{k=0}^{M} \binom{M}{k} (-1)^k (M-k)^{n-j} - \sum_{l=0}^{M} \binom{M}{l} (-1)^l (M-l)^n$$

of which the second summation is nothing but the $j = 0$ contribution of the first sum. Eliminating these terms we end up with the r.h.s. of expression (10), hereby proving Theorem 1.

Since the m variables X_i are independent, the mean and variance as well as the moment generating function of $X = \sum_{i=1}^{m} X_i$ can easily be derived. The mean is given by:

$$\mu_X = \sum_{i=1}^{m} \mu_{X_i} = \frac{m\lambda e^{\lambda}}{e^{\lambda} - 1} \tag{12}$$

and the variance by:

$$\sigma_X^2 = \sum_{i=1}^{m} \sigma_{X_i}^2 = \frac{m\lambda e^{\lambda}}{e^{\lambda} - 1} \left[1 - \frac{\lambda}{e^{\lambda} - 1}\right] \tag{13}$$

while the moment generating function is given by:

$$M_X(t) = \prod_{i=1}^{m} M_{X_i}(t) = \left[\frac{e^{\lambda e^t} - 1}{e^{\lambda} - 1}\right]^m \tag{14}$$

From Theorem 1, we may derive that the cumulative distribution function for the random variable $X = \sum_{i=1}^{m} X_i$ is given by:

$$F_X(x) = \begin{cases} 0 & \text{if } x < m \\ \displaystyle\sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{(e^{\lambda} - 1)^m} \left[e_{\lfloor x \rfloor}(\lambda(m-k)) \right. & \\ \qquad\qquad \left. - e_{m-1}(\lambda(m-k))\right] & \text{if } x \geqslant m \end{cases} \tag{15}$$

4 The sum of non-identical zero-truncated Poisson distributed random variables

In this section, we extend the results of the previous section to the case in which the random variables X_i are no longer identical, meaning that each X_i is characterized by a parameter λ_i.
We will follow a constructive approach, elaborating the cases $m = 2$ and $m = 3$, after which a general pattern can be derived leading to the result for any value of m.

For $m = 2$ and $n \in I\!N; n \geqslant 2$, we find that the probability function may be written as:

$$\begin{aligned}
P[X_1 + X_2 = n] &= \sum_{j=1}^{n-1} P[X_1 = j]P[X_2 = n-j] \\
&= \sum_{j=1}^{n-1} \frac{\lambda_1^j}{j!(e^{\lambda_1}-1)} \cdot \frac{\lambda_2^{n-j}}{(n-j)!(e^{\lambda_2}-1)} \\
&= \frac{1}{n!(e^{\lambda_1}-1)(e^{\lambda_2}-1)} \sum_{j=1}^{n-1} \binom{n}{j} \lambda_1^j \lambda_2^{n-j} \\
&= \frac{(\lambda_1+\lambda_2)^n - \lambda_1^n - \lambda_2^n}{n!(e^{\lambda_1}-1)(e^{\lambda_2}-1)}
\end{aligned}$$

In a similar way, we find the following result for the probability function when $m = 3$ and $n \in I\!N; n \geqslant 3$:

$$\begin{aligned}
P[X_1 + X_2 + X_3 = n] &= \sum_{i_1=1}^{n-2} \sum_{i_2=1}^{n-1-i_1} P[X_1 = i_1]P[X_2 = i_2]P[X_3 = n-i_1-i_2] \\
&= \sum_{i_1=1}^{n-2} \sum_{i_2=1}^{n-1-i_1} \frac{\lambda_1^{i_1}}{i_1!(e^{\lambda_1}-1)} \cdot \frac{\lambda_2^{i_2}}{i_2!(e^{\lambda_2}-1)} \cdot \frac{\lambda_3^{n-i_1-i_2}}{(n-i_1-i_2)!(e^{\lambda_3}-1)} \\
&= \sum_{i_1=1}^{n-2} \sum_{i_2=1}^{n-1-i_1} \binom{n}{i_1, i_2, n-i_1-i_2} \frac{\lambda_1^{i_1}\lambda_2^{i_2}\lambda_3^{n-i_1-i_2}}{n!(e^{\lambda_1}-1)(e^{\lambda_2}-1)(e^{\lambda_3}-1)} \\
&= \sum_{i_1=0}^{n} \sum_{i_2=0}^{n-i_1} \binom{n}{i_1, i_2, n-i_1-i_2} \frac{\lambda_1^{i_1}\lambda_2^{i_2}\lambda_3^{n-i_1-i_2}}{n!(e^{\lambda_1}-1)(e^{\lambda_2}-1)(e^{\lambda_3}-1)} \\
&\quad - \sum_{i_2=0}^{n} \binom{n}{i_2} \frac{\lambda_2^{i_2}\lambda_3^{n-i_2}}{n!(e^{\lambda_1}-1)(e^{\lambda_2}-1)(e^{\lambda_3}-1)} \\
&\quad - \sum_{i_1=0}^{n} \binom{n}{i_1} \frac{\lambda_1^{i_1}\lambda_3^{n-i_1}}{n!(e^{\lambda_1}-1)(e^{\lambda_2}-1)(e^{\lambda_3}-1)} \\
&\quad - \sum_{i_1=0}^{n} \binom{n}{i_1} \frac{\lambda_1^{i_1}\lambda_2^{n-i_1}}{n!(e^{\lambda_1}-1)(e^{\lambda_2}-1)(e^{\lambda_3}-1)} \\
&\quad + \frac{\lambda_1^n + \lambda_2^n + \lambda_3^n}{n!(e^{\lambda_1}-1)(e^{\lambda_2}-1)(e^{\lambda_3}-1)} \\
&= \frac{(\lambda_1+\lambda_2+\lambda_3)^n - (\lambda_1+\lambda_2)^n - (\lambda_1+\lambda_3)^n - (\lambda_2+\lambda_3)^n}{n!(e^{\lambda_1}-1)(e^{\lambda_2}-1)(e^{\lambda_3}-1)} \\
&\quad + \frac{\lambda_1^n + \lambda_2^n + \lambda_3^n}{n!(e^{\lambda_1}-1)(e^{\lambda_2}-1)(e^{\lambda_3}-1)}
\end{aligned}$$

In the same manner, one may derive for the case $m = 4$ and $n \in I\!N; n \geqslant 4$:

$$
\begin{aligned}
P[\sum_{i=1}^{4} X_i = n] = {} & [(\lambda_1+\lambda_2+\lambda_3+\lambda_4)^n - (\lambda_1+\lambda_2+\lambda_3)^n \\
& -(\lambda_1+\lambda_2+\lambda_4)^n - (\lambda_1+\lambda_3+\lambda_4)^n - (\lambda_2+\lambda_3+\lambda_4)^n \\
& +(\lambda_1+\lambda_2)^n + (\lambda_1+\lambda_3)^n + (\lambda_1+\lambda_4)^n + (\lambda_2+\lambda_3)^n \\
& +(\lambda_2+\lambda_4)^n + (\lambda_3+\lambda_4)^n - \lambda_1^n - \lambda_2^n - \lambda_3^n - \lambda_4^n] \,/ \\
& \left[n!(e^{\lambda_1}-1)(e^{\lambda_2}-1)(e^{\lambda_3}-1)(e^{\lambda_4}-1)\right]
\end{aligned}
\tag{16}
$$

Hence, a clear pattern is appearing. In order to write the generalisation with corresponding proof, we need to introduce the following notation:

- $S_m(p) = \{\{i_1, \ldots, i_{m-p}\} |\ i_j \in \{1, \ldots, m\}$, with $i_j < i_k$ if $j < k\}$ i.e. the set of all possible ordered lists of indices between 1 and m with length $m - p$;
- $S^+_{m+1}(p) = \{\{i_1, \ldots, i_{m-p}, m+1\} |\ i_j \in \{1, \ldots, m\}$, with $i_j < i_k$ if $j < k\}$ i.e. the set of all possible ordered lists of indices between 1 and $m+1$ with length $m+1-p$ in which the last element equals $m+1$;
- $S^-_{m+1}(p) = \{\{i_1, \ldots, i_{m-p}, i_{m+1-p}\} |\ i_j \in \{1, \ldots, m\}$, with $i_j < i_k$ if $j < k\}$ i.e. the set of all possible ordered lists of indices between 1 and m with length $m+1-p$;
- $\sigma(p)$ stands for such an ordered list of length m in which p elements have been dropped, i.e. $\sigma(p) \in S_m(p)$;
- $\sum\limits_{\sigma(p) \in S_m(p)}$ stands for the sum over all possible ordered lists of length $m - p$ of indices between 1 and m;
- $\sigma(p)_j$ is the j^{th} component in the list of length $m - p$

It should be remarked that $S_{m+1}(p)$ is the direct sum of the sets $S^+_{m+1}(p)$ and $S^-_{m+1}(p)$, $\forall p = 0, \ldots m+1$, with $S^-_{m+1}(0) = \emptyset$. It is also clear from the above definitions that $S^-_{m+1}(p+1) = S_m(p)$. Finally, it must be noticed that the number of elements in the set $S_m(p)$ is given by $\#S_m(p) = \binom{m}{p}$.
Let us first prove the following theorem, which may be seen as a generalisation of formula (11).

Theorem 2 *$\forall l = 0, \ldots, m-2$ with $2 \leqslant m \in I\!N$:*

$$\sum_{p=0}^{m-1} (-1)^p \sum_{\sigma(p) \in S_m(p)} \left(\sum_{j=1}^{m-p} \lambda_{\sigma(p)_j}\right)^{m-1-l} \equiv 0 \tag{17}$$

<u>*Proof:*</u> The proof will be based on an inductive structure.

<u>Case $m = 2$</u>: $\Rightarrow l = 0$: $\sum_{p=0}^{1} \sum_{\sigma(p) \in S_2(p)} (\sum_{j=1}^{2-p} \lambda_{\sigma(p)_j})^{1-l} = (\lambda_1 + \lambda_2) - \lambda_1 - \lambda_2 \equiv 0$

We are well aware of its redundancy in the proof itself, though for sake of clarity we still elaborate one more case in order to stress that relation (17) must be valid for every $l = 0, \ldots, m-2$.

<u>Case $m = 3$</u>: $\Rightarrow l = 0, 1$: $\sum_{p=0}^{2} \sum_{\sigma(p) \in S_3(p)} (\sum_{j=1}^{3-p} \lambda_{\sigma(p)_j})^{2-l}$

$$\underline{l=0}: \quad \sum_{p=0}^{2} \sum_{\sigma(p) \in S_3(p)} \left(\sum_{j=1}^{3-p} \lambda_{\sigma(p)_j}\right)^2 = (\lambda_1 + \lambda_2 + \lambda_3)^2 - \big[(\lambda_1 + \lambda_2)^2 + (\lambda_1 + \lambda_3)^2 + (\lambda_2 + \lambda_3)^2\big] + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \equiv 0$$

$$\underline{l=1}: \quad \sum_{p=0}^{2} \sum_{\sigma(p) \in S_3(p)} \left(\sum_{j=1}^{3-p} \lambda_{\sigma(p)_j}\right) = (\lambda_1 + \lambda_2 + \lambda_3) - [(\lambda_1 + \lambda_2) + (\lambda_1 + \lambda_3) + (\lambda_2 + \lambda_3)] + \lambda_1 + \lambda_2 + \lambda_3 \equiv 0$$

Now suppose that the theorem is valid for $m = M$, $\forall l = 0, \ldots, M-2$. We then have to prove that it also holds for $m = M+1$, $\forall l = 0, \ldots M-1$, i.e.

$$\sum_{p=0}^{M} (-1)^p \sum_{\sigma(p) \in S_{M+1}(p)} \left(\sum_{j=1}^{M+1-p} \lambda_{\sigma(p)_j}\right)^{M-l} \equiv 0, \ \forall l = 0, \ldots M-1 \tag{18}$$

The left-hand side of this expression may be written as:

$$
\begin{aligned}
\text{l.h.s.} &= \sum_{p=0}^{M}(-1)^p \sum_{\sigma(p)\in S^+_{M+1}(p)} (\sum_{j=1}^{M+1-p} \lambda_{\sigma(p)_j})^{M-l} \\
&\qquad + \sum_{p=1}^{M}(-1)^p \sum_{\sigma(p)\in S^-_{M+1}(p)} (\sum_{j=1}^{M+1-p} \lambda_{\sigma(p)_j})^{M-l} \\
&= \sum_{p=0}^{M-1}(-1)^p \sum_{\sigma(p)\in S_M(p)} (\sum_{j=1}^{M-p} \lambda_{\sigma(p)_j} + \lambda_{M+1})^{M-l} + (-1)^M \lambda_{M+1}^{M-l} \\
&\qquad - \sum_{q=0}^{M-1}(-1)^q \sum_{\sigma(q)\in S_M(q)} (\sum_{j=1}^{M-q} \lambda_{\sigma(q)_j})^{M-l} \\
&= \sum_{p=0}^{M-1}(-1)^p \sum_{\sigma(p)\in S_M(p)} \left[\sum_{k=0}^{M-l}\binom{M-l}{k}\lambda_{M+1}^k(\sum_{j=1}^{M-p} \lambda_{\sigma(p)_j})^{M-l-k}\right] \\
&\qquad - \sum_{q=0}^{M-1}(-1)^q \sum_{\sigma(q)\in S_M(q)} (\sum_{j=1}^{M-q} \lambda_{\sigma(q)_j})^{M-l} + (-1)^M \lambda_{M+1}^{M-l}
\end{aligned}
$$

The summation over q is nothing but the $k = 0$ contribution in the first summation. Furthermore the $k = M - l$ contribution may be written as:

$$
\sum_{p=0}^{M-1}(-1)^p \sum_{\sigma(p)\in S_M(p)} \lambda_{M+1}^{M-l} = \sum_{p=0}^{M-1}(-1)^p \binom{M}{p}\lambda_{M+1}^{M-l} = -(-1)^M \lambda_{M+1}^{M-l} \tag{19}
$$

Hence, the l.h.s. reduces to

$$
\begin{aligned}
\text{l.h.s.} &= \sum_{k=1}^{M-1-l}\binom{M-l}{k}\lambda_{M+1}^k \left[\sum_{p=0}^{M-1}(-1)^p \sum_{\sigma(p)\in S_M(p)} (\sum_{j=1}^{M-p} \lambda_{\sigma(p)_j})^{M-l-k}\right] \\
&= \sum_{r=0}^{M-2-l}\binom{M-l}{r+1}\lambda_{M+1}^{r+1} \underbrace{\left[\sum_{p=0}^{M-1}(-1)^p \sum_{\sigma(p)\in S_M(p)} (\sum_{j=1}^{M-p} \lambda_{\sigma(p)_j})^{M-1-l-r}\right]}_{\substack{\equiv 0 \\ \forall l = 0,\ldots,M-2 \\ \forall r = 0,\ldots,M-2-l}}
\end{aligned}
$$

Using the induction hypothesis one may conclude that the expression between brackets is equal to zero. This enables us to conclude that expression (18) is satisfied, hereby proving theorem 2.

We now present the following theorem for the density function of the sum of m independent non-identical zero truncated Poisson variables:

Theorem 3 *Let X_i ($i = 1, \ldots, m$) be zero-truncated Poisson distributed random variables with respective parameter λ_i, and $X = \sum_{i=1}^{m} X_i$. The probability function for the sum X is given by:*

$$P[X = n] = \begin{cases} \dfrac{\displaystyle\sum_{p=0}^{m-1} (-1)^p \sum_{\sigma(p)\in S_m(p)} (\sum_{j=1}^{m-p} \lambda_{\sigma(p)_j})^n}{\displaystyle n! \prod_{i=1}^{m} (e^{\lambda_i} - 1)} & \textit{if } m \leqslant n \in I\!N \\ 0 & \textit{elsewhere} \end{cases} \tag{20}$$

<u>*Proof:*</u> First we use the independence of the random variables X_i:

$$\begin{aligned} P[X = n] &= \sum_{k=1}^{n-m} P[\sum_{i=1}^{m} X_i = n - k] P[X_{m+1} = k] \\ &= \frac{1}{\displaystyle n! \prod_{i=1}^{m+1} (e^{\lambda_i} - 1)} \sum_{k=1}^{n-m} \binom{n}{k} \lambda_{m+1}^k \sum_{p=0}^{m-1} (-1)^p \sum_{\sigma(p)\in S_m(p)} (\sum_{j=1}^{m-p} \lambda_{\sigma(p)_j})^{n-k} \end{aligned}$$

This should be proven to be equal to

$$P[X = n] = \frac{1}{\displaystyle n! \prod_{i=1}^{m+1} (e^{\lambda_i} - 1)} \sum_{p=0}^{m} (-1)^p \sum_{\sigma(p)\in S_{m+1}(p)} (\sum_{j=1}^{m+1-p} \lambda_{\sigma(p)_j})^n \tag{21}$$

Canceling the common factor $\left[n! \displaystyle\prod_{i=1}^{m+1} (e^{\lambda_i} - 1) \right]^{-1}$ it suffices to prove that

$$\begin{aligned} &\sum_{p=0}^{m} (-1)^p \sum_{\sigma(p)\in S_{m+1}(p)} (\sum_{j=1}^{m+1-p} \lambda_{\sigma(p)_j})^n \\ &\qquad = \sum_{k=1}^{n-m} \binom{n}{k} \lambda_{m+1}^k \sum_{p=0}^{m-1} (-1)^p \sum_{\sigma(p)\in S_m(p)} (\sum_{j=1}^{m-p} \lambda_{\sigma(p)_j})^{n-k} \end{aligned} \tag{22}$$

Expanding the l.h.s. of this equation, we obtain:

$$
\begin{aligned}
\text{l.h.s.} &= \sum_{p=0}^{m}(-1)^p \sum_{\sigma(p)\in S^+_{m+1}(p)} (\sum_{j=1}^{m+1-p} \lambda_{\sigma(p)_j})^n \\
&\qquad\qquad\qquad + \sum_{p=1}^{m}(-1)^p \sum_{\sigma(p)\in S^-_{m+1}(p)} (\sum_{j=1}^{m+1-p} \lambda_{\sigma(p)_j})^n \\
&= \sum_{p=0}^{m-1}(-1)^p \sum_{\sigma(p)\in S_m(p)} (\sum_{j=1}^{m-p} \lambda_{\sigma(p)_j} + \lambda_{m+1})^n + (-1)^m \lambda_{m+1}^n \\
&\qquad\qquad\qquad - \sum_{q=0}^{m-1}(-1)^q \sum_{\sigma(q)\in S_m(q)} (\sum_{j=1}^{m-q} \lambda_{\sigma(q)_j})^n \\
&= \sum_{p=0}^{m-1}(-1)^p \sum_{\sigma(p)\in S_m(p)} \left[\sum_{k=0}^{n} \binom{n}{k} \lambda_{m+1}^k (\sum_{j=1}^{m-p} \lambda_{\sigma(p)_j})^{n-k}\right] + (-1)^m \lambda_{m+1}^n \\
&\qquad\qquad\qquad - \sum_{q=0}^{m-1}(-1)^q \sum_{\sigma(q)\in S_m(q)} (\sum_{j=1}^{m-q} \lambda_{\sigma(q)_j})^n
\end{aligned}
$$

The $k=0$ and $k=n$ contributions in the former expression cancel with the sum over q and the term $(-1)^m \lambda_{m+1}^n$ respectively. Hence, the l.h.s. reduces to

$$
\text{l.h.s.} = \sum_{k=1}^{n} \binom{n-1}{k} \lambda_{m+1}^k \sum_{p=0}^{m-1}(-1)^p \sum_{\sigma(p)\in S_m(p)} (\sum_{j=1}^{m-p} \lambda_{\sigma(p)_j})^{n-k} \tag{23}
$$

which should be proven to be equal to the r.h.s. of eq. (22). Thus, it suffices to prove that

$$
\sum_{k=n-m+1}^{n-1} \binom{n}{k} \lambda_{m+1}^k \sum_{p=0}^{m-1}(-1)^p \sum_{\sigma(p)\in S_m(p)} (\sum_{j=1}^{m-p} \lambda_{\sigma(p)_j})^{n-k} \equiv 0
$$

or

$$
\sum_{l=0}^{m-2} \binom{n}{m-1-l} \lambda_{m+1}^{n+1-m+l} \sum_{p=0}^{m-1}(-1)^p \sum_{\sigma(p)\in S_m(p)} (\sum_{j=1}^{m-p} \lambda_{\sigma(p)_j})^{m-1-l} \equiv 0 \tag{24}
$$

which, on account of theorem 2, is satisfied. Hence, this proves theorem 3.

As the $X_i's$ are independent, the mean and variance of $X = \sum_{i=1}^m X_i$ are given by

$$
\mu_X = \sum_{i=1}^{m} \mu_{X_i} = \sum_{i=1}^{m} \frac{\lambda_i e^{\lambda_i}}{e^{\lambda_i} - 1} \tag{25}
$$

and

$$\sigma_X^2 = \sum_{i=1}^{m} \sigma_{X_i}^2 = \sum_{i=1}^{m} \frac{\lambda_i e^{\lambda_i}}{e^{\lambda_i} - 1} \left[1 - \frac{\lambda_i}{e^{\lambda_i} - 1} \right] \tag{26}$$

while the moment generating function is given by:

$$M_X(t) = \prod_{i=1}^{m} M_{X_i}(t) = \prod_{i=1}^{m} \left[\frac{e^{\lambda_i e^t} - 1}{e^{\lambda_i} - 1} \right]. \tag{27}$$

In order to determine the cumulative distribution of $X = \sum_{i=1}^{m} X_i$, we start from the definition:

$$F_X(x) = \sum_{n \leqslant \lfloor x \rfloor} P[X = n] = \sum_{n=m}^{\lfloor x \rfloor} P[X = n] \tag{28}$$

if $x \geqslant m$, otherwise $F_X(x) = 0$.
Now using expression (20) we obtain:

$$F_X(x) = \sum_{n=m}^{\lfloor x \rfloor} \frac{\displaystyle\sum_{p=0}^{m-1} (-1)^p \sum_{\sigma(p) \in S_m(p)} (\sum_{j=1}^{m-p} \lambda_{\sigma(p)_j})^n}{\displaystyle n! \prod_{i=1}^{m} (e^{\lambda_i} - 1)} \tag{29}$$

which on account of theorem 2 may be rewritten as:

$$F_X(x) = \sum_{n=1}^{\lfloor x \rfloor} \frac{\displaystyle\sum_{p=0}^{m-1} (-1)^p \sum_{\sigma(p) \in S_m(p)} (\sum_{j=1}^{m-p} \lambda_{\sigma(p)_j})^n}{\displaystyle n! \prod_{i=1}^{m} (e^{\lambda_i} - 1)} \tag{30}$$

or

$$F_X(x) = \frac{\displaystyle\sum_{p=0}^{m-1} (-1)^p \sum_{\sigma(p) \in S_m(p)} \sum_{n=0}^{\lfloor x \rfloor} \frac{1}{n!} (\sum_{j=1}^{m-p} \lambda_{\sigma(p)_j})^n + (-1)^m}{\displaystyle\prod_{i=1}^{m} (e^{\lambda_i} - 1)} \tag{31}$$

Hence, we finally obtain the following expression for the cumulative distribution for the sum of m independent zero-truncated Poisson random variables:

$$F_X(x) = \begin{cases} 0 & \text{if } x < m \\ \dfrac{\displaystyle\sum_{p=0}^{m-1}(-1)^p \sum_{\sigma(p)\in S_m(p)} e_{\lfloor x \rfloor}(\sum_{j=1}^{m-p} \lambda_{\sigma(p)_j}) + (-1)^m}{\displaystyle\prod_{i=1}^{m}(e^{\lambda_i} - 1)} & \text{if } x \geqslant m \end{cases} \tag{32}$$

The expressions (20) and (32) are of course respective generalizations of expressions (9) and (15). This can easily be shown in the former case by considering all $\lambda_{\sigma(p)_j}$ and λ_i in relation (20) equal to a unique λ. We then have that:

$$(\sum_{j=1}^{m-p} \lambda_{\sigma(p)_j})^n = (\sum_{j=1}^{m-p} \lambda)^n = (m-p)^n \lambda^n \tag{33}$$

and, referring to our previous remarks, that:

$$\sum_{\sigma(p)\in S_m(p)} (\sum_{j=1}^{m-p} \lambda_{\sigma(p)_j})^n = \sum_{\sigma(p)\in S_m(p)} (m-p)^n \lambda^n = \binom{m}{p} (m-p)^n \lambda^n \tag{34}$$

Taking the above relation into account and reducing the expression $\prod_{i=1}^m (e^{\lambda_i} - 1)$ to $(e^{\lambda} - 1)^m$ the expression (9) is obtained.

5 Conclusion

In this paper, we have derived expressions for the density function and cumulative distribution function of the sum of m independent, either identical or non-identical zero-truncated Poisson random variables. To the best of our knowledge, this paper is the first one to present such expressions. Though the work presented here is largely theoretical, we are confident that it will further prove its usefulness in our future research, which aims at integrating the inventory models of the operations management literature with the purchase quantity models available from the marketing research field. In view of this objective, the presented work provides a necessary first step.

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